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FINITE DIMENSIONAL INTEGRABLE SYSTEMS

Supersymmetric Construction of Exactly Solvable Potentials and Nonlinear Algebras*

G. Junker and P. Roy[1)]

Institut für Theoretische Physik, Universität Erlangen-Nürnberg, Staudtstrasse 7, D 91058 Erlangen, Germany

Abstract—Using algebraic tools of supersymmetric quantum mechanics, we construct classes of conditionally exactly solvable potentials being the supersymmetric partners of the linear or radial harmonic oscillator. With the aid of the raising and lowering operators of these harmonic oscillators and the SUSY operators, we construct ladder operators for these new conditionally solvable systems. It is found that these ladder operators, together with the Hamilton operator, form a nonlinear algebra, which is of the quadratic and cubic types for the SUSY partners of the linear and radial harmonic oscillators, respectively.

1. INTRODUCTION, SUMMARY, AND OUTLOOK

Over the last decade, supersymmetric (SUSY) quantum mechanics has become an important tool in various branches of theoretical physics. By way of example, we indicate that, in quantum mechanical problems, SUSY has been found to be a very useful algebraic tool [1]. In particular, the class of exactly solvable quantum systems has been enlarged by such methods [2]. Quite recently, these methods have even been extended to constructing conditionally exactly solvable problems [3], where, in addition, it has been shown that these systems have a nonlinear algebraic structure.

The objective of this study is to generalize the approach given in [3] to a much wider class of conditionally exactly solvable systems being the SUSY partners of the linear or radial harmonic oscillator. In doing this, we will first review the basic tools of SUSY quantum mechanics [1] that we are going to use. In Section 3, we will present in some detail the general construction principle that we proposed in [3]. In Section 4, we present the results for a linear harmonic oscillator. Section 5 and 6 contain our results on a radial harmonic oscillator with unbroken and broken SUSY, respectively.

In addition to constructing conditionally exactly solvable problems, we also analyze their algebraic structure, which proves to be uniquely characterized by their SUSY partner—that is, we obtain a quadratic algebra for the SUSY partners of a linear oscillator and find a cubic algebra for a radial oscillator (with unbroken as well as broken SUSY).

As becomes clear from our general method in Section 3, the present approach can also be applied to other shape-invariant SUSY systems such as the radial hydrogen atom and a Morse or a Pöschl–Teller oscillator. Another application consists in constructing exactly solvable drift potentials associated with the Fokker–Planck equation. As matter of fact, this has already been done for a harmonic oscillator by Hongler and Zheng [4].

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1) Physics and Applied Mathematics Unit, Indian Statistical Institute, Calcutta, 700035 India.

2. SUPERSYMMETRIC QUANTUM MECHANICS

Witten's model of supersymmetric quantum mechanics involves a pair of standard Schrödinger Hamiltonians

$$H_{\pm} = -\frac{1}{2}\frac{d^2}{dx^2} + V_{\pm}(x) \qquad (1)$$

acting on the Hilbert space $\mathcal{H}$ of square-integrable functions on the configuration space M, which we will assume to be the real line in the case of the linear harmonic oscillator, $\mathcal{H} = L^2(\mathbb{R})$, or the positive half-line in the case of the radial harmonic oscillator, $\mathcal{H} = \{\psi \in L^2(\mathbb{R}^+) | \psi(0) = 0\}$. The so-called SUSY partner potentials

$$V_{\pm}(x) = \frac{1}{2}(W^2(x) \pm W'(x)) \qquad (2)$$

are given by the SUSY potential $W : M \longrightarrow \mathbb{R}$ and by its derivative $W' = dW/dx$. In terms of the SUSY operators

$$A = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + W(x)\right), \quad A^{\dagger} = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + W(x)\right), \qquad (3)$$

the SUSY partner Hamiltonians are given by $H_+ = AA^{\dagger} \geq 0$ and $H_- = A^{\dagger}A \geq 0$.

With the aid of the operators in (3), it is easy to show that H_+ and H_- are essentially isospectral. To be more

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explicit, we denote the eigenfunctions and eigenvalues of $H_\pm$ by $\psi_n^\pm$ and $E_n^\pm$, respectively; that is,

$$H_\pm \psi_n^\pm(x) = E_n^\pm \psi_n^\pm(x), \quad n = 0, 1, 2, \dots. \tag{4}$$

In the case of unbroken SUSY (here we will use the convention [1] that the zero-energy eigenstate of the SUSY system belongs to H_-), the ground state of H_- is characterized by the relations

$$E_0^- = 0, \quad \psi_0^-(x) = C\exp\{-\int dx W(x)\} \in \mathcal{H} \tag{5}$$

with C being a proper normalization constant. The remaining spectrum of H_- coincides with the complete spectrum of H_+, and the corresponding eigenfunctions are related by the SUSY transformations

$$\begin{aligned} E_{n+1}^- &= E_n^+ > 0, \\ \psi_{n+1}^-(x) &= (E_n^+)^{-1/2} A^\dagger \psi_n^+(x), \\ \psi_n^+(x) &= (E_{n+1}^-)^{-1/2} A \psi_{n+1}^-(x). \end{aligned} \tag{6}$$

In the case of broken SUSY, H_+ and H_- are strictly isospectral, and the eigenfunctions are also related by the SUSY transformations

$$\begin{aligned} E_n^- &= E_n^+ > 0, \\ \psi_n^-(x) &= (E_n^+)^{-1/2} A^\dagger \psi_n^+(x), \\ \psi_n^+(x) &= (E_n^-)^{-1/2} A \psi_n^-(x). \end{aligned} \tag{7}$$

Although the above relations (6) and (7) are valid in the cases of continuous spectra as well, we consider here only systems having a purely discrete spectrum.

From relations (5) and (6) or (7), it is obvious that, knowing the spectral properties of, say, H_+, we can immediately obtain the complete spectral properties of the SUSY partner Hamiltonian H_-. It is the fact that is our basis for constructing (conditionally) exactly solvable potentials, by which we mean that the eigenvalues and eigenfunctions of the corresponding Schrödinger Hamiltonian can be given in an explicit closed form (under certain conditions imposed on the potential parameters). Furthermore, the SUSY operators (3) also allow us to construct, from known ladder operators of H_+, the corresponding ladder operators for H_-, which prove to close a nonlinear algebra.

3. CONSTRUCTING EXACTLY SOLVABLE POTENTIALS

In this section, we present our basic idea underlying the construction of (conditionally) exactly solvable potentials. As was anticipated in the last section, the basic idea consists in choosing the SUSY potential W such that the partner potential V_+ becomes one of the well-known exactly solvable ones—that is, the eigenvalue problem for the corresponding Hamiltonian H_+ is exactly solvable. In this way, we can eventually find (through a proper choice of W) new partner potentials that are also exactly solvable; that is, the spectral properties for H_- are obtainable via the SUSY transformations (6) or (7).

In order to find an appropriate class of SUSY potentials, we employ the ansatz [1]

$$W(x) = \Phi(x) + f(x), \tag{8}$$

where Φ is a so-called shape-invariant SUSY potential [1]—that is, for $f \equiv 0$, the corresponding partner potentials $V_\pm$ belong to a known class of exactly solvable ones. For a nonvanishing f, we have

$$\begin{aligned} V_+(x) = \frac{1}{2}[&\Phi^2(x) + \Phi'(x) + f^2(x) \\ &+ 2\Phi(x)f(x) + f'(x)]. \end{aligned} \tag{9}$$

If we now choose f such that it obeys the generalized Riccati equation

$$f^2(x) + 2\Phi(x)f(x) + f'(x) = 2(\varepsilon - 1), \tag{10}$$

at least under certain conditions on the parameters contained in Φ and for certain values of $\varepsilon \in \mathbb{R}$, the two partner potentials are given by

$$V_+(x) = \frac{1}{2}\Phi^2(x) + \frac{1}{2}\Phi'(x) + \varepsilon - 1, \tag{11}$$

$$V_-(x) = \frac{1}{2}\Phi^2(x) - \frac{1}{2}\Phi'(x) - f'(x) + \varepsilon - 1. \tag{12}$$

Clearly, the potential V_+ is, by construction, shape-invariant and is therefore exactly solvable. Via the SUSY transformation, we can also solve the eigenvalue problem for H_- associated with the above potential V_-, which, in view of our assumption that the potential parameters must take certain values, is sometimes referred to as a conditionally exactly solvable potential [5]. A first and obvious condition on the parameter ε is that it must be sufficiently large to give rise to a strictly positive Hamiltonian $H_+ > 0$. If this were not the case, the SUSY transformations would lead to "wave functions" that do not belong to the Hilbert space $\mathcal{H}$. This, for example, may happen if the solution f to (10) contains a singularity in the configuration space M. Note that the SUSY operators (3) are given by

$$\begin{aligned} A &= \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + \Phi(x) + f(x)\right), \\ A^\dagger &= \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + \Phi(x) + f(x)\right) \end{aligned} \tag{13}$$

and that they must leave the Hilbert space invariant: $A: \mathcal{H} \longrightarrow \mathcal{H}, A^\dagger: \mathcal{H} \longrightarrow \mathcal{H}$.

To find such regular solutions to equation (10), we linearize it by setting $f(x) = u'(x)/u(x)$, which leads to an ordinary, homogeneous, and linear second-order differential equation

$$u''(x) + 2\Phi(x)u'(x) + 2(1-\varepsilon)u(x) = 0. \quad (14)$$

In terms of u, the conditionally exactly solvable potential then has the form

$$\begin{aligned} V_-(x) &= \frac{1}{2}\Phi^2(x) - \frac{1}{2}\Phi'(x) \\ &+ \frac{u'(x)}{u(x)}\left(2\Phi(x) + \frac{u'(x)}{u(x)}\right) - \varepsilon + 1. \end{aligned} \quad (15)$$

Ensuring the regularity of f now amounts to obtaining the most general solutions to equation (14) that are (without loss of generality) strictly positive on M. This is equivalent to requiring that V_- not have any additional singularities apart from that of V_+. The latter may only exist at $x = 0$ for the case of $M = \mathbb{R}^+$.

In the following, we will consider three examples corresponding to a linear harmonic oscillator with unbroken SUSY and a radial harmonic oscillator both with unbroken and with broken SUSY. For these systems, we also know how to construct ladder operators, which close a linear algebra. With the aid of the SUSY operators (13), we are then able to obtain ladder operators for the conditionally exactly solvable system H_-, which prove to close a nonlinear algebra.

4. LINEAR HARMONIC OSCILLATOR

As a first example, we consider the SUSY potential of a linear harmonic oscillator on the real line $M = \mathbb{R}$:

$$\Phi(x) = x. \quad (16)$$

It is straightforward to verify that, in this case, the potential (11) is indeed that of a linear harmonic oscillator,

$$V_+(x) = \frac{1}{2}x^2 + \varepsilon - \frac{1}{2}, \quad (17)$$

whose energy eigenvalues and eigenfunctions are

$$\begin{aligned} E_n^+ &= n + \varepsilon, \\ \psi_n^+(x) &= [\sqrt{\pi}2^n n!]^{-1/2} H_n(x)\exp\{-x^2/2\}, \end{aligned} \quad (18)$$

where H_n stands for Hermite polynomials of orders $n = 0, 1, 2, \ldots$. Since a strictly positive spectrum is required for H_+, we arrive at a first condition on the parameter ε: $\varepsilon > 0$.

Let us now consider a solution to (14) with the linear SUSY potential (16). Upon the substitution $z = -x^2$, this differential equation transforms into that for the confluent hypergeometric function; therefore, the most general solution has the form

$$\begin{aligned} u(x) &= \alpha_1F_1\left(\frac{1-\varepsilon}{2}, \frac{1}{2}, -x^2\right) + \beta x_1F_1\left(\frac{2-\varepsilon}{2}, \frac{3}{2}, -x^2\right) \\ &= e^{-x^2}\left[\alpha_1F_1\left(\frac{\varepsilon}{2}, \frac{1}{2}, x^2\right) + \beta x_1F_1\left(\frac{1+\varepsilon}{2}, \frac{3}{2}, x^2\right)\right], \end{aligned} \quad (19)$$

$\alpha, \beta \le R$ being two additional parameters of the system. Since we seek strictly positive solutions, the real parameter α must not vanish; hence, it can be set to unity without loss of generality. In addition, the real parameter β must obey the inequality $|\beta| < 2\Gamma\left(\frac{1+\varepsilon}{2}\right)/\Gamma\left(\frac{\varepsilon}{2}\right)$, which follows from the positivity condition $u > 0$ via the asymptotic form

$$u(x) = x^{\varepsilon-1}\left(\frac{\Gamma(1/2)}{\Gamma(\varepsilon/2)} + \beta\frac{\Gamma(3/2)}{\Gamma\left(\frac{1+\varepsilon}{2}\right)}\right)[1 + O(1/x)]. \quad (20)$$

Note that, for $\beta = 0$, the positivity requirement on u leads to $\varepsilon > 0$, a condition that has already been obtained above from the positivity of H_+. Under these conditions, the potential (15) is given by

$$V_-(x) = \frac{1}{2}x^2 - \varepsilon + \frac{1}{2} + \frac{u'(x)}{u(x)}\left(2x + \frac{u'(x)}{u(x)}\right), \quad (21)$$

which is now a conditionally exactly solvable potential. A plot of this potential for $0 < \varepsilon \le 3$ and $\beta = 0$ is displayed in Fig. 1. For small ε and $\beta = 0$, the potential V_- exhibits two deep and one shallow minimum located at the origin. In fact, the parameter ε is the tunneling splitting due to the tunnel effect between two deep minima. For large values of ε, the shallow minimum at the center $x = 0$ becomes deeper, and the other two minima, which are located symmetrically about the origin, disappear. For nonvanishing β, the basic structure of V_- is the same, but, now, it is no longer symmetric with respect to $x = 0$.

For the above potential (21), the ground-state energy eigenvalue and eigenfunction of H_- are given by

$$E_0^- = 0, \quad \psi_0^-(x) = \frac{C}{u(x)}\exp\{-x^2/2\}. \quad (22)$$

Note that, because of (20), the above ground-state wave function is square-integrable; therefore, SUSY is unbroken. The remaining spectral properties of H_- fol-

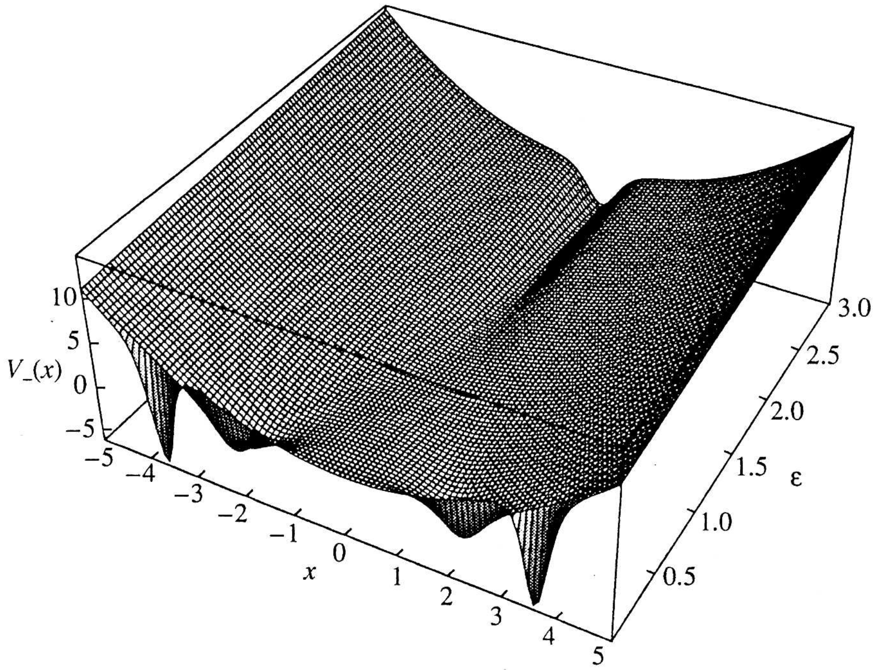

Fig. 1. Family of SUSY partner potentials that is given by (21) and which corresponds to the linear-harmonic-oscillator potential (17). Here, we show only the symmetric case of $\beta = 0$.

low from those of H_+ via the SUSY transformation (6):

$$E^-_{n+1} = n + \varepsilon,$$

$$\psi^-_{n+1}(x) = \frac{\exp\{-x^2/2\}}{[\sqrt{\pi}2^{n+1}n!(n+\varepsilon)]^{1/2}} \times\left(H_{n+1}(x) + H_n(x)\frac{u'(x)}{u(x)}\right). \quad (23)$$

It is worth noting that, for $\beta = 0$ and for an odd integer $\varepsilon = 2N + 1 > 0$, the solution given by (19) becomes a polynomial in x^2 of degree N with no real zeros—that is, $u(x) = (1 + g_1x^2)\ldots(1 + g_Nx^2)$ with $g_i > 0$. These cases—in particular, those for $N = 1$ and 2—have been discussed in [3] (see, however, also [6] for a different approach to such cases and for their connection to nonlinear superalgebras).

Let us now construct ladder operators for H_-. In doing this, we first recall the well-known ladder operators for a linear harmonic oscillator specified by H_+. We have

$$a = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right). \quad (24)$$

These operators obey the linear algebra

$$[H_+, a] = -a, \quad [H_+, a^\dagger] = a^\dagger, \quad [a, a^\dagger] = 1 \quad (25)$$

and act on the eigenstates of H_+ as follows:

$$a\psi^+_n(x) = \sqrt{n}\psi^+_{n-1}(x), \qquad a^\dagger\psi^+_n(x) = \sqrt{n+1}\psi^+_{n+1}(x). \quad (26)$$

With the aid of the SUSY operators (13), we can now construct similar ladder operators [3] for the SUSY partner H_-:

$$B = A^\dagger a A, \quad B^\dagger = A^\dagger a^\dagger A. \quad (27)$$

Obviously, these operators act as lowering and raising operators:

$$B\psi^-_{n+1}(x) = \sqrt{E^-_{n-1}nE^-_{n+1}}\psi^-_n(x), \qquad B^\dagger\psi^-_{n+1}(x) = \sqrt{E^+_{n+1}(n+1)E^-_{n+1}}\psi^-_{n+2}(x). \quad (28)$$

However, the ground state remains isolated; that is, $B\psi^-_0(x) = 0 = B^\dagger\psi^-_0(x)$. By using these relations, we can easily verify that, together with the Hamiltonian H_-, the ladder operators B and $B^\dagger$ close the nonlinear algebra

$$[H_-, B] = -B, \quad [H_-, B^\dagger] = B^\dagger, \qquad [B, B^\dagger] = 3H_-^2 - (2\varepsilon - 1)H_-, \quad (29)$$

which is of the quadratic type. Owing to unbroken SUSY, which implies that $H_-\psi^-_0(x) = 0$, this algebra is defined on the full Hilbert space $\mathcal{H} = L^2(\mathbb{R})$.

5. RADIAL HARMONIC OSCILLATOR WITH UNBROKEN SUSY

As a second example, we consider the SUSY potential

$$\Phi(x) = x - \frac{\gamma+1}{x}, \quad \gamma \ge 0, \quad (30)$$

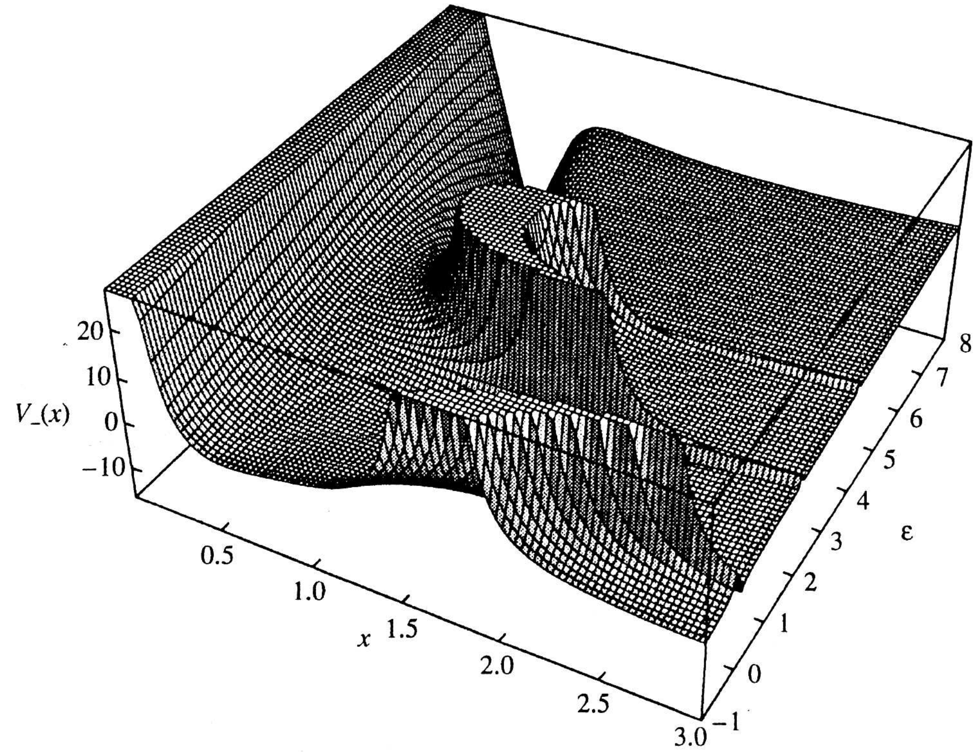

Fig. 2. Family of SUSY partner potentials (35) corresponding to the radial-harmonic-oscillator class (31). Here, we show only the cases of $\beta = 0$ and $\gamma = 1$. Because of condition (34), the allowed ranges of ε are $0 < \varepsilon < 2$ and $4 < \varepsilon < \infty$. For the forbidden regions, the figure clearly displays singularities in V_-.

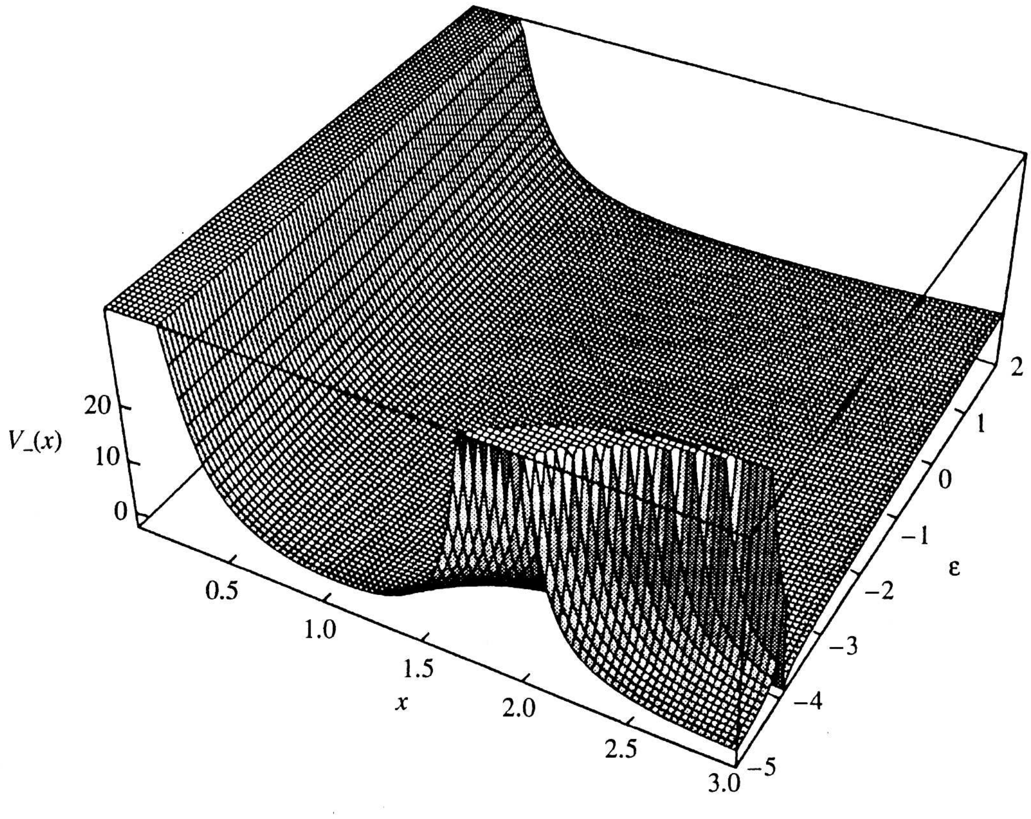

Fig. 3. Family of SUSY partner potentials (47) corresponding to the radial-harmonic-oscillator class (44) with broken SUSY. Here, we show only the case of $\gamma = 1$. The allowed range of ε is $-4 < \varepsilon$.

which in turn gives rise to the radial-harmonic-oscillator potential

$$V_+(x) = \frac{x^2}{2} + \frac{(\gamma+1)(\gamma+2)}{2x^2} + \varepsilon - \gamma - \frac{3}{2}. \tag{31}$$

The energy eigenvalues and eigenfunctions of the corresponding Hamiltonian H_+ are

$$E_n^+ = 2n + 1 + \varepsilon,$$
$$\psi_n^+(x) = \left[\frac{2n!}{\Gamma(n+\gamma+5/2)}\right]^{1/2} x^{\gamma+2} L_n^{\gamma+3/2}(x^2) e^{-x^2/2}, \tag{32}$$

where L_n^ν stands for a Laguerre polynomial of degree n [7]. The positivity of H_+ leads to the first restriction: $\varepsilon > -1$.

Let us now consider positive solutions to equation (14). We have

$$u(x) = {}_1F_1\left(\frac{1-\varepsilon}{2}, -\gamma - \frac{1}{2}, -x^2\right)$$
$$+ \beta x^{2\gamma+3} {}_1F_1\left(2 + \gamma - \frac{\varepsilon}{2}, \frac{5}{2} + \gamma, -x^2\right), \tag{33}$$

where we have already set $\alpha = 1$ without loss of generality. The positivity of the above solution amounts to imposing the following conditions on the parameters β, γ, and ε:

$$0 < \frac{\Gamma\left(-\gamma - \frac{1}{2}\right)}{\Gamma(\varepsilon/2 - \gamma - 1)},$$
$$|\beta| < \frac{\Gamma\left(-\gamma - \frac{1}{2}\right)}{\Gamma(\varepsilon/2 - \gamma - 1)} \frac{\Gamma\left(\frac{1+\varepsilon}{2}\right)}{\Gamma(5/2 + \gamma)}. \tag{34}$$

The corresponding partner potential has the form

$$V_-(x) = \frac{x^2}{2} + \frac{\gamma(\gamma+1)}{2x^2} - \gamma - \varepsilon$$
$$+ \frac{1}{2} + \frac{u'(x)}{u(x)}\left(2x - 2\frac{\gamma+1}{x} + \frac{u'(x)}{u(x)}\right). \tag{35}$$

By virtue of the above conditions, it is now also a conditionally exactly solvable potential. Figure 2 shows this potential for $\beta = 0$, $\gamma = 1$, and $-1 < \varepsilon \le 8$. Note that, for $\varepsilon \le 0$ and $2 \le \varepsilon \le 4$, the potential (35) exhibits singularities, as might have been expected, because these values of ε are not allowed for $\gamma = 1$. Since SUSY remains unbroken for all the allowed values of the parameters, the ground-state energy of the SUSY partner Hamiltonian H_- vanishes, and the corresponding eigenstate is obtained from (5). The remaining spectral properties of H_- are found via the SUSY transformations (6):

$$E_0^- = 0, \quad \psi_0^-(x) = \frac{C}{u(x)} x^{\gamma+1} e^{-x^2/2},$$
$$E_{n+1}^- = 2n + 1 + \varepsilon,$$
$$\psi_{n+1}^-(x) = \frac{1}{\sqrt{4n + 2 + 2\varepsilon}} \times\left(-\frac{d}{dx} + x - \frac{\gamma+1}{x} + \frac{u'(x)}{u(x)}\right)\psi_n^+(x). \tag{36}$$

To construct the ladder operators for H_-, we recall the corresponding operators for a radial harmonic oscillator [7], which in essence are built from those given in (24):

$$c = a^2 - \frac{(\gamma+1)(\gamma+2)}{2x^2},$$
$$c^\dagger = (a^\dagger)^2 - \frac{(\gamma+1)(\gamma+2)}{2x^2}. \tag{37}$$

These operators act on the eigenstates of H_+ as

$$c\psi_n^+(x) = -2\sqrt{n(n+\gamma+3/2)}\psi_{n-1}^+(x),$$
$$c^\dagger\psi_n^+(n) = -2\sqrt{(n+1)(n+\gamma+5/2)}\psi_{n+1}^+(x), \tag{38}$$

and, as in the preceding example, close a linear Lie algebra:

$$[H_+, c] = -2c, \quad [H_+, c^\dagger] = 2c^\dagger,$$
$$[c, c^\dagger] = 4(H_+ + \gamma - \varepsilon + 3/2). \tag{39}$$

Furthermore, they also allow us to construct ladder operators for the quantum system characterized by H_-:

$$D = A^\dagger c A, \quad D^\dagger = A^\dagger c^\dagger A. \tag{40}$$

These operators act on the eigenstates of H_- in the following way:

$$D\psi_{n+1}^-(x) = -2\sqrt{E_{n-1}^+ n(n+\gamma+3/2)E_{n+1}^-}\psi_n^-(x),$$
$$D^\dagger\psi_{n+1}^-(x)$$
$$= -2\sqrt{E_{n+1}^+(n+1)(n+\gamma+5/2)E_{n+1}^-}\psi_{n+2}^-(x),$$
$$D\psi_0^-(x) = 0 = D^\dagger\psi_0^-(x). \tag{41}$$

The last line shows that the ground state is again isolated, which follows from the fact that SUSY is unbroken. With the aid of the above relations, we can verify that these operators, together with the Hamiltonian, close the nonlinear algebra

$$[H_-, D] = -2D, \quad [H_-, D^\dagger] = 2D^\dagger,$$
$$[D, D^\dagger] = 8H_-^3 + 12(\gamma - \varepsilon + 3/2)H_-^2 \qquad (42)$$
$$- 4(2\varepsilon\gamma - \varepsilon^2 + 3\varepsilon - 1)H_-,$$

which is of the cubic type.

6. RADIAL HARMONIC OSCILLATOR WITH BROKEN SUSY

So far, we have considered only examples with unbroken SUSY. However, a radial harmonic oscillator also admits a broken SUSY, in which case the second term in (30) is opposite in sign. Hence, we consider the SUSY potential [1]

$$\Phi(x) = x + \frac{\gamma + 1}{x}, \quad \gamma \geq 0, \qquad (43)$$

which yields the radial-harmonic-oscillator potential

$$V_+(x) = \frac{x^2}{2} + \frac{\gamma(\gamma + 1)}{2x^2} + \varepsilon + \gamma + \frac{1}{2} \qquad (44)$$

and the following spectral properties of the corresponding Hamiltonian H_+:

$$E_n^+ = 2n + 2\gamma + 2 + \varepsilon,$$
$$\psi_n^+(x) = \left[\frac{2n!}{\Gamma(n + \gamma + 3/2)}\right]^{1/2} x^{\gamma + 1} L_n^{\gamma + 1/2}(x^2) e^{-x^2/2}. \qquad (45)$$

Clearly, we have the condition $-2 - 2\gamma < \varepsilon$. This condition is identical to that obtained from the positivity of the following solution to (14):

$$u(x) = {}_1F_1\left(\frac{1 - \varepsilon}{2}, \gamma + \frac{3}{2}, -x^2\right). \qquad (46)$$

Note that the second linearly independent solution to equation (14) is not allowed ($\beta = 0$) in order for SUSY to remain broken. The corresponding partner potential has the form

$$V_-(x) = \frac{x^2}{2} + \frac{(\gamma + 1)(\gamma + 2)}{2x^2} + \gamma - \varepsilon$$
$$+ \frac{3}{2} + \frac{u'(x)}{u(x)}\left(2x + 2\frac{\gamma + 1}{x} + \frac{u'(x)}{u(x)}\right), \qquad (47)$$

and the spectral properties of the associated Hamiltonian H_- are immediately obtained from (7):

$$E_n^- = 2n + 2\gamma + 2 + \varepsilon,$$
$$\psi_n^-(x) = \frac{1}{\sqrt{4n + 4\gamma + 4 + 2\varepsilon}} \qquad (48)$$
$$\times\left(-\frac{d}{dx} + x + \frac{\gamma + 1}{x} + \frac{u'(x)}{u(x)}\right)\psi_n^+(x).$$

Figure 3 shows the potential (47) for $\gamma = 1$ and $-5 \leq \varepsilon \leq 2$.

As before, we can introduce the ladder operators $D = A^\dagger c A$ and $D^\dagger = A^\dagger c^\dagger A$, which obey the nonlinear algebra

$$[H_-, D] = -2D, \quad [H_-, D^\dagger] = 2D^\dagger,$$
$$[D, D^\dagger] = 8H_-^3 - 12(\gamma + \varepsilon + 1/2)H_-^2 \qquad (49)$$
$$+ 4(2\varepsilon\gamma + \varepsilon^2 + \varepsilon + 1)H_-.$$

This algebra can also be obtained from the unbroken SUSY case (42) by replacing γ by $-\gamma - 2$. In contrast to the case of unbroken SUSY, however, the ladder operators act here on all eigenstates of H_-. In other words, the ground state is not isolated. In fact, we have the relation

$$\psi_n^-(x) = \left(-\frac{1}{4}\right)^n$$
$$\times\left[n!\left(\gamma + \frac{3}{2}\right)_n\left(\gamma + 1 + \frac{\varepsilon}{2}\right)_n\left(\gamma + 2 + \frac{\varepsilon}{2}\right)_n\right]^{-1/2}(D^\dagger)^n\psi_0^-(x), \qquad (50)$$

where the ground-state wave function is

$$\psi_0^-(x) = \frac{x^{\gamma + 1}\exp\{-x^2/2\}}{\sqrt{(2\gamma + \varepsilon + 2)\Gamma\left(\gamma + \frac{3}{2}\right)}}$$
$$\times\left(2x - \gamma - 1 + \frac{\gamma + 1}{x} + \frac{u'(x)}{u(x)}\right). \qquad (51)$$

A discussion for the special case of $\varepsilon = 3$ and arbitrary γ is given in [3].

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